Imperial College London

ICIC Imperial Centre for Inference & Cosmology

Gravitational Wave Backgrounds

By Ciarán Conneely. Supervisor Prof. Andrew Jaffe.

The direct detection of gravitational waves in 2015 by Advanced LIGO [1] has started a new era of astronomy and we assume that there are sufficient sources to incoherently sum to an all-pervasive **background** which can be studied statistically.

These waves have many properties that are **analogous to photons** – especially in terms of their **polarisation**. Both types of wave have two polarisations: "**plus**" and "**cross**" for gravitational waves and "**vertical**" and "**horizontal**" for photons. However, these are coordinate system dependent and so analysis of photon polarisation backgrounds (such as measurements of the **Cosmic Microwave Background** (CMB)) is done in terms of coordinate independent "**E**" and "**B**" **modes**. In an almost mathematically identical way, a gravitational wave background can be considered in terms of similar modes.

This project considers the **relative strength** of these modes and the **relative sensitivity** of detectors to them.

CMB Style Decompositions

The gravitational wave background can either be written in terms of + and x polarisations or decomposed using spherical tensor harmonics into E and B modes as below [2]

$$h_{\mu\nu}(f,\hat{k}) = H_+(f,\hat{k})h^+_{\mu\nu}(\hat{k}) + H_\times(f,\hat{k})h^\times_{\mu\nu}(\hat{k})$$
$$= \sum_{(lm)} \left[a^E_{(lm)}(f)Y^E_{(lm)\mu\nu}(\hat{k}) + a^B_{(lm)}(f)Y^B_{(lm)\mu\nu}(\hat{k})\right]$$

Using these modes, and in analogy to CMB techniques, we can define the measured power spectra for each mode

$$\hat{C}^P_l = \frac{1}{2l+1}\sum_m |a^P_{(lm)}|^2 \quad P \in \{E, B\}$$

where the strengths of each mode are not necessarily equal. If they are not equal then this indicates something special about the background.

$h_{\mu\nu}(f,\hat{k})$	Gravitational wave perturbation tensor – the strength of the signal.
$h^{+,\times}_{\mu\nu}(\hat{k})$	+ and x basis tensors.
$H_{+,\times}(f,\hat{k})$	Strength of + and x polarisations.
$Y^{E,B}_{(lm)\mu\nu}(\hat{k})$	Spherical tensor harmonics – a basis for tensor fields on the sphere.
$a^{E,B}_{(lm)}(f)$	Coefficients for spherical tensor harmonic expansion.
$Y_{\pm 2,lm}(\hat{k})$	Spin-weighted spherical harmonics – a basis for spin-$\pm$2 fields on the sphere.
$a_{\pm 2,lm}(f)$	Coefficients for spin-weighted spherical harmonic expansion.
$Q,U(f,\hat{k})$	Stokes Parameters.

Comparisons of CMB and Gravitational Wave E and B Modes

Both sets of E and B modes can be calculated in terms of spin-$\pm$2 weighted spherical harmonics.

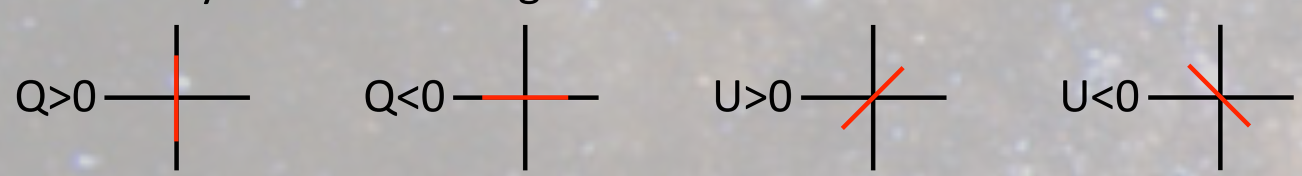
$$a^E_{lm}(f) = -\frac{1}{2}(a_{2,lm}(f) + a_{-2,lm}(f)) \qquad a^B_{lm}(f) = \frac{i}{2}(a_{2,lm}(f) - a_{-2,lm}(f))$$

where these spin-$\pm$2 harmonics are calculated in similar ways for CMB polarisation and gravitational waves.

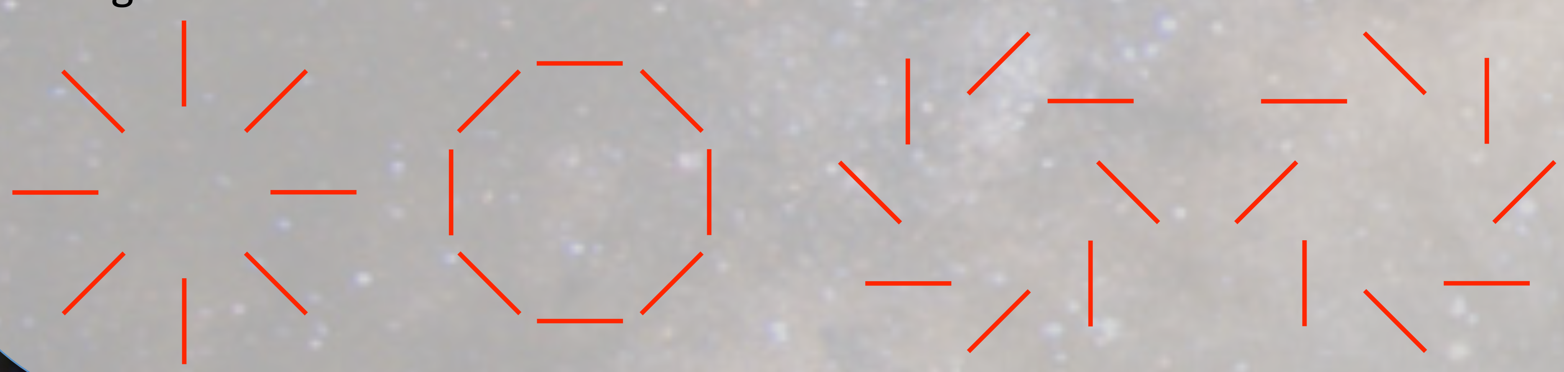
CMB

$$a^{EM}_{\pm 2,lm}(f) = \int (Q(f,\hat{k}) \pm iU(f,\hat{k}))_{\pm 2}Y^*_{lm}(\hat{k})\mathrm{d}^2\hat{k}$$

Q and U are two of the Stokes parameters for electromagnetic waves and E and B modes are usually visualised using these:



Giving **E modes** **B Modes**



Gravitational Waves

$$a^{GW}_{\pm 2,lm}(f) = \int (H_+(f,\hat{k}) \pm iH_\times(f,\hat{k}))_{\pm 2}Y^*_{lm}(\hat{k})\mathrm{d}^2\hat{k}$$

H_+ and $H_\times$ can be represented similarly but are not affect by a sign change and so 2 are used in the visualisation of E and B modes rather than 4:

Giving **E mode** **B Mode**

Examples

Power Spectra for Galactic White Dwarf Binaries

Galactic white dwarf binaries are one known stochastic background. Consider a toy model for the galactic distribution where the probability of a white dwarf binary being present decreases with distance from the galactic centre (decreasing more rapidly with galactic latitude than galactic longitude) and simulate it – Fig. 1. Each binary is given randomised parameters [3] and the power spectra computed – Fig. 2.

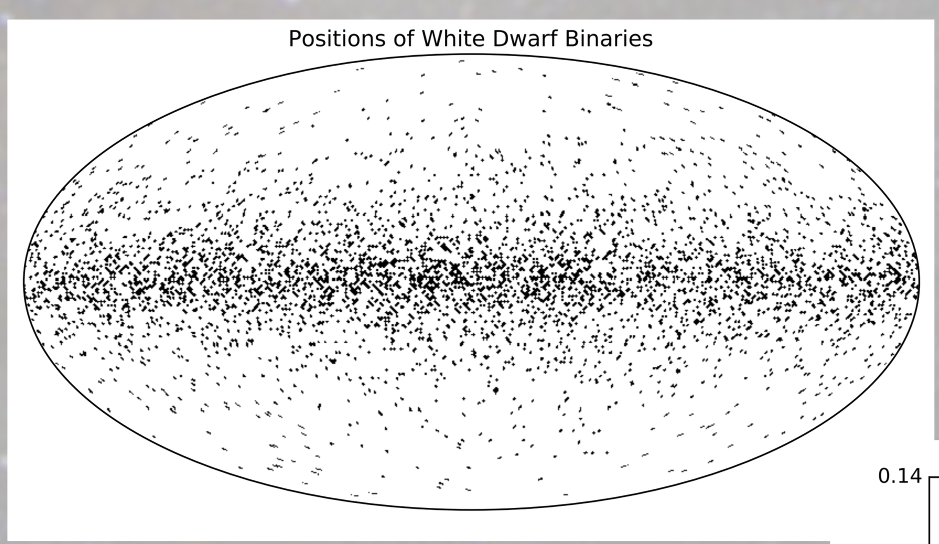


Fig. 1
Positions of white dwarf binaries.

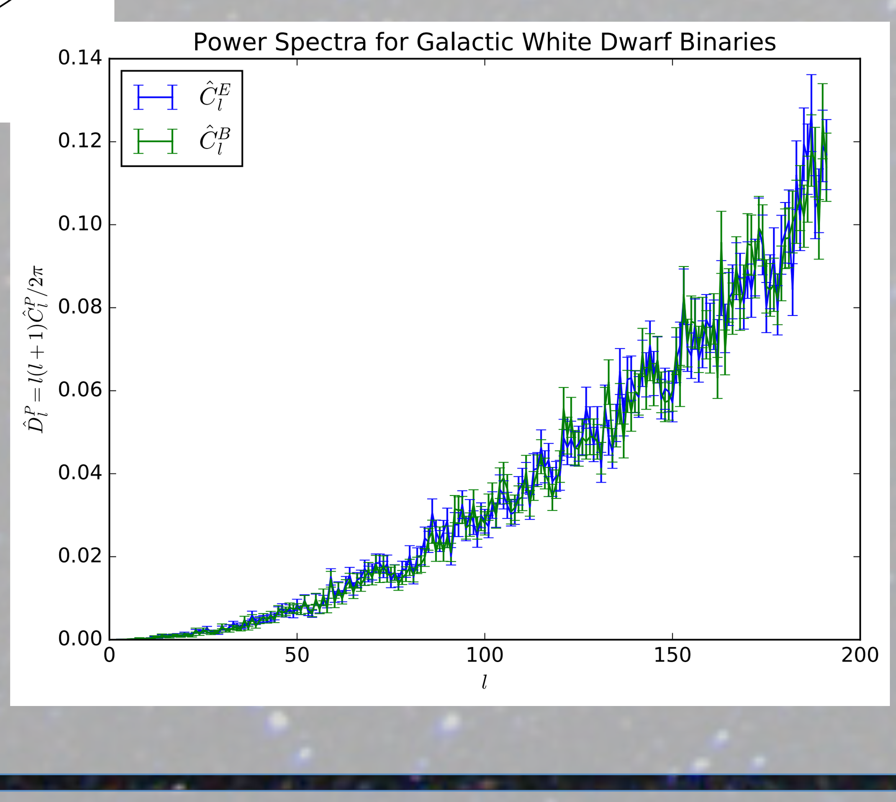


Fig. 2
Power spectra of a distribution of white dwarfs.
Because this is a toy model the amplitudes of gravitational wave sources are not realistic and so neither are the magnitudes of $\hat{D}^P_l$.

Power Spectra for a Cosmic String Loop

A single cosmic string loop viewed from a particular direction can produce pure + polarised gravitational waves [4]. The E and B power spectra are computed in Fig. 3 for a single + polarised source in the z-direction.

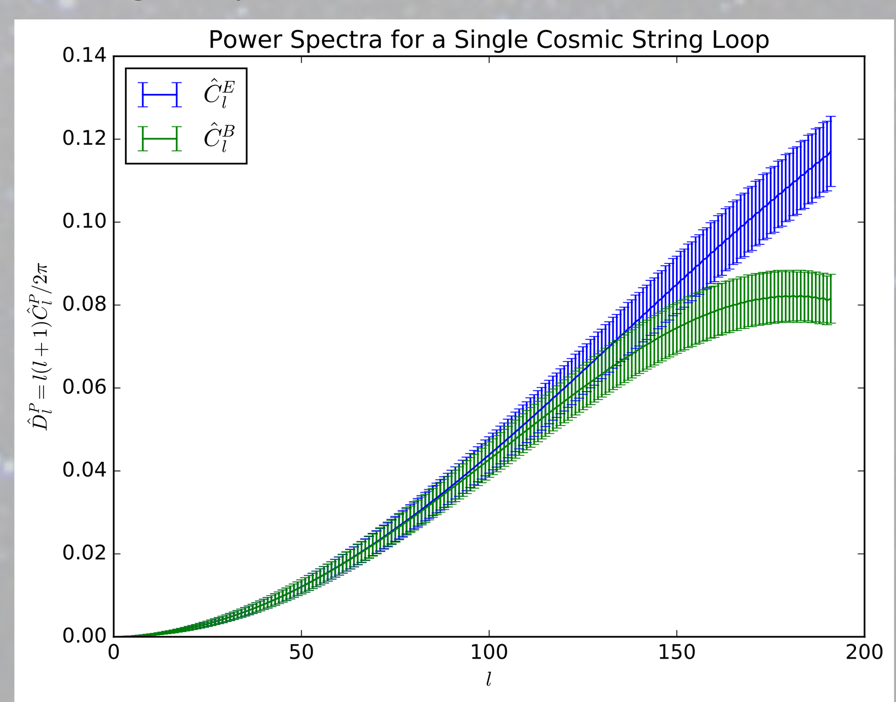


Fig. 3
Power spectrum of a single + polarised source in the z-direction.
Again, source strength is arbitrary so the magnitudes of $\hat{D}^P_l$ are not realistic.

Conclusions

- For the white dwarf background, the E and B power spectra are statistically equal.
- For the single + polarised source (i.e. representing a single cosmic string loop) the E and B spectra only diverge at small scales close to the pixel scale. This makes sense because the source is small and so on large scales there should be no significant difference.

This last point is for a single loop with a particular orientation. An unaligned, stochastically distributed ensemble of loops would have equal strength E and B modes.

References:
[1] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. Physical Review Letters, 2016. [2] J. Gair, J. Romano, and S. Taylor. Mapping gravitational-wave backgrounds using methods from CMB analysis: Application to pulsar timing arrays. Physical Review D, 2014. [3] S. E. Timpano, L. J. Rubbo, and N. J. Cornish. Characterizing the galactic gravitational wave background with LISA. Physics Review D, 2006. [4] T. Vachaspati and A. Vilenkin, Gravitational radiation from cosmic strings. Physical Review D, 1985.
Picture - Credits: A. Fujii. http://www.nasa.gov/topics/earth/features/2012-alignment.html